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A MODIFIED METHOD FOR SOLVING THE ASYMMETRIC TRAFFIC EQUILIBRIUM PROBLEM

by

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Abstract. In this paper we employ a restriction (simplicial decomposition) strategy to effectively implement a modified Newton method to solve the general asymmetric network equilibrium problem. Implementation details are discussed. Numerical results on one small-scale and one medium-scale problem, with varying asymmetry levels, are presented.

1. Introduction.

Equilibrium traffic assignment has progressively become an integral part of planning software, due to rapid advances in algorithms capable of solving large-scale nonlinear network flow problems. Recently, it has been recognized that network equilibrium problems of very general nature can be cast into a nonlinear complementarity or variational inequality framework, thus enabling the easy integration of features such as production curves, demand markets with asymmetric cross-elasticities, interactions between the links of the network, and several transportation modes. In the general case, such models do not admit equivalent convex mathematical programming formulations. To deal with this fundamentally novel situation, two main approaches have been investigated:

1. The iterative approach , whereby the original problem is replaced by a sequence of easier-to-solve problems, whose solutions sequence converges to the solution of the original problem. This is the rationale behind projection methods (Dafermos [5], Bertsekas and Gafni [4]), the diagonalization algorithm (Florian and Spiess [7]) and linear approximation or Newton-related methods (Josephy [10],[11], Pang and Yu [16]).

2. The optimization approach , where the problem is formulated as a mathematical program rather than a fixed point problem. In Nguyen and Dupuis [15] the objective is linear, but there are infinitely many non-network constraints. In Marcotte [14], the objective is neither differentiable nor convex, whereas in Dussault and Marcotte [6] the objective and

the constraints are nonconvex.

In this paper, we discuss the application of the modified Newton method proposed by Dussault and Marcotte [6] to the fixed demand traffic equilibrium problem. Under the usual monotonicity and strict complementarity assumptions, global convergence is assured. If furthermore, the cost function is strongly monotone, then the asymptotic rate of convergence is quadratic. Also, the algorithm is robust with respect to nonsymmetric perturbations to the cost function.

The rest of the paper is organized as follows: first we state the problem and describe briefly the algorithm; then we give a detailed discussion of its implementation using the restricted simplicial decomposition strategy proposed by Lawphongpanich and Hearn [12]; finally, numerical results are reported on one small-scale and one medium-scale network, with varying asymmetry levels. A proof of quadratic convergence of the algorithm when applied to the non-strongly monotone master program is provided in the first appendix.

2. Statement of the problem.

Let us first give a precise definition of the problem, using the following notation:

$G = (N,A)$: strongly connected network, where N is a node-set of cardinality n and A an arc-set of cardinality m;

Ω : set of origin nodes of cardinality M;

Δ : set of destination nodes;

v_a^l : flow on arc a associated with origin $l \in \Omega$ ("commodity flow");

$v^l = (v_a^l)_{a \in A}$: flow vector associated with origin $l \in \Omega$;

$v = (v^l)_{l \in \Omega}$: multicommodity flow vector;

$w_a = \sum_{l \in \Omega} v_a^l$: total flow on arc a;

$w = (w_a)_{a \in A}$: total flow vector;

$H_a(w)$: nonnegative cost on link a, possibly dependent on flows other than w_a;

$H(w) = (H_a(w))_{a \in A}$: link cost function (might represent traversal time or generalized cost);

$F(v) = (H^t(w), \ldots, H^t(w))^t$: "multicommodity" cost function;

C : node-arc incidence matrix of the network;

$$B = \begin{pmatrix} C & & & \\ & C & 0 & \\ & & \ddots & \\ & 0 & & C \end{pmatrix}$$

b^l : vector of displacements associated with origin (commodity) $l \in \Omega$; b_j^l represents the trip demand between nodes $l \in \Omega$ and $j \in A$, and can be nonzero only if j is a destination node, i.e. $j \in \Delta$;

$b = (b^l)_{l \in \Omega}$: overall displacement vector, sometimes referred to as the origin-destination matrix;

Φ : feasibility polyhedron, i.e.

$$\Phi = \{ v \mid Cv^l = b^l,\ v^l \geq 0,\ l \in \Omega \} = \{ v \mid Bv = b,\ v \geq 0 \}$$

According to the preceding notations, $v^* = (v^{*l})_{l \in \Omega}$ is an equilibrium multicommodity flow vector if and only if it is feasible and satisfies the variational inequality:

$$F^t(v^*)(v^* - v) \leq 0 , \qquad \text{for all } v \text{ in } \Phi \tag{1}$$

Remarks.

1. As observed by Smith [17], the variational inequality formulation of the traffic assignment problem is a mathematical formulation of Wardrop's user equilibrium principle [18] which states that all utilized paths have equal costs, and those costs are inferior or equal to the costs of non-utilized paths. This is usually referred to as an equilibrium flow.

2. Since there are no explicit link-capacity constraints in the model, the linear problem with constant link costs:

$$\min_{y \in \Phi} F^t(v)y$$

reduces to L single-origin, single-commodity problems, and can therefore be solved by computing L shortest path trees rooted at the origin nodes.

3. Since $F(v)$ is nonnegative, one needs not consider cycles. Thus Φ may be assumed bounded, hence compact. This insures the existence of at least one solution to the traffic equilibrium problem (see Aashtiani and Magnanti [1]).

Following Hearn [9], we define the gap function g associated with the variational inequality (1) as:

$$g(v) = \underset{y\in\Phi}{\text{Max}}\ F^t(v)(v-y) \tag{2}$$

Thus v satisfies the variational inequality (1) if and only if it is a zero of the function g. From the nonnegativity of g, it follows that the equilibrium solutions are global minimizers for the nonconvex, nondifferentiable mathematical programming problem:

$$\underset{v\in\Phi}{\text{Min}}\ \{\ g(v) = \underset{y\in\Phi}{\text{Max}}\ F^t(v)(v-y)\ \} \tag{3}$$

for which a globally convergent algorithm has been proposed in Marcotte [14].

In the following section, we present a modified Newton method based on the reformulation of (3) as a <u>differentiable</u>, nonconvex but unimodal mathematical program.

Throughout the paper, we will assume that the mapping F is continuously differentiable and monotone on Φ, i.e.:

$$(y-v)^t F'(v)(y-v) \geq 0 \quad , \quad \text{for all } y,v \text{ in } \Phi$$

where $F'(v)$ is the Jacobian matrix of F evaluated at v. This monotonicity assumption implies that the set of equilibria is convex.

Finally we will use the following definition of strict complementarity:

Definition (strict complementarity).

Let T denote the minimal face containing the set S^* of equilibrium flow vectors. Then strict complementarity holds for the variational inequality (1) if and only if we have that $F^t(x)(x-y)$ is strictly negative whenever x is in S^* and y is in Φ-T.

3. A modified Newton algorithm.

From (3), it follows that any equilibrium flow vector v must be optimal for the linear programming problem:

$$\min_{y \in \Phi} \; F^t(v)y$$

From linear programming duality theory, it follows that v, together with an optimal dual vector α, must satisfy the equality-inequality system:

$$\begin{aligned} (F(v) + B^t\alpha)^t v &= 0 \\ F(v) + B^t\alpha &\geq 0 \qquad (4) \\ v &\in \Phi \end{aligned}$$

When F is the gradient of some function $f: \Phi \longrightarrow R^1$, (4) are the Kuhn-Tucker necessary and sufficient optimality conditions for the convex minimization problem:

$$\min_{v \in \Phi} \; f(v)$$

This equivalence has been first recognized by Beckmann [3] for the special case where F is link-separable.

We write next the complementarity system (4) as a standard mathematical program:

$$\begin{aligned} &\underset{v,\alpha}{\text{Min}} \ (F(v) + B^t\alpha)^t v \\ &\text{s.t.} \quad F(v) + B^t\alpha \geq 0 \\ &\qquad\qquad v \in \Phi \end{aligned}$$

Thus, noting that Bv=b, it follows that it is equivalent to:

$$\begin{aligned} &\underset{v,\alpha}{\text{Min}} \ F^t(v)v + b^t\alpha \\ &\text{s.t.} \quad F(v) + B^t\alpha \geq 0 \\ &\qquad\qquad v \in \Phi \end{aligned} \tag{5}$$

Remark. There exists a simple network interpretation of problem (5) in path-flow space. Consider an assignment problem with one origin-destination pair and a demand of one. Let $x=(x_1,\ldots,x_p)^t$ denote a path-flow vector and $R=(R_1(x),\ldots,R_p(x))^t$ denote the corresponding cost function. Then the equivalent of (5) in path-flow space is:

$$\begin{aligned} &\underset{v,\alpha}{\text{Min}} \ R^t(x)x + \gamma \\ &\text{s.t.} \quad R_i(x) \geq -\gamma \quad , \quad i=1,\ldots,p \\ &\qquad \sum_{i=1}^{p} x_i = 1 \\ &\qquad\qquad x \geq 0 \end{aligned} \tag{6}$$

$R^t(x)x$ represents the total or average transportation cost and, for given x, the objective of (6) is minimized if $-\gamma = \underset{i}{\text{Min}}\, R_i(x) = -\gamma^*$, the cost of

the minimum cost path. Thus $R^t(x)x + \gamma^*$ is the gap associated with path-flow x. At an equilibrium x^*, the objective is zero, and the average cost is equal to the shortest path cost. This is clearly a restatement of Wardrop's user equilibrium principle for this simple problem. The analysis readily extends to multiple origins and destinations.

Problem (5) can be solved by any general purpose mathematical programming package. However we will see how the straightforward application of a linear approximation algorithm yields, given the special nature of the problem, a quadratically converging procedure. In the description, the letters LA stand for Linear Approximation.

Algorithm A.

0. Initialization.

Let $v^1 \in \Phi$ and $\alpha^1 \in \arg \operatorname{Min}_{\alpha} b^t\alpha$

$$\text{s.t. } F(v^1) + B^t\alpha \geq 0;$$

$k \leftarrow 1$.

1. Find descent direction.

Let $\begin{vmatrix} LA(v^k) \\ LA(\alpha^k) \end{vmatrix}$ be an optimal solution to the linear problem:

$$\operatorname{Min}_{v,\alpha} \; [F(v^k) + F'^t(v^k)v^k]^t v + b^t\alpha$$

$$\text{s.t. } F(v^k) + F'(v^k)(v-v^k) + B^t\alpha \geq 0 \qquad (7)$$

$$v \in \Phi.$$

2. Perform arc search (line search).

$$\text{Let } \vartheta_k \in \arg \min_{0 \le \vartheta \le 1} g[(1-\vartheta)v^k + \vartheta LA(v^k)].$$

3. Update.

$$v^{k+1} \leftarrow (1-\vartheta_k)v^k + \vartheta_k LA(v^k);$$

$$\alpha^{k+1} \in \arg \min_{\alpha} b^t\alpha \quad \text{s.t.} \quad F(v^{k+1}) + B^t\alpha \ge 0;$$

$k \leftarrow k+1$ and return to 1.

At step 2 of algorithm A, we minimize the gap function (2) rather than the objective of (5). Consequently, the constraints of (5) are always satisfied in the minimization process, hence the expression "arc search". In Dussault and Marcotte [6] it is shown that $LA(v^k)$ is a descent direction for g at v^k. Also, due to the relation

$$g(v) = \min_{\alpha} F^t(v)v + b^t\alpha \quad \text{s.t.} \quad F(v) + B^t\alpha \ge 0$$

the value of the dual vector α^{k+1} associated with v^{k+1} is a byproduct of the evaluation of g at v^{k+1}, and can therefore be obtained at no additional cost.

The main results concerning algorithm A are summarized in the following theorem:

Theorem.

Let (v^k) be a sequence generated by algorithm A. Then we have:

1. $\lim_{k\to\infty} g(v^k) = 0$ (global convergence).

2. $g(v^{k+1}) < g(v^k)$ whenever v^k is not an equilibrium solution (monotone convergence).

3. If F is affine, finite convergence occurs.

If in addition the mapping F is strongly monotone, and strict complementarity holds[1] at the solution, then:

4. The optimal face T (set of active constraints at the solution) is implicitly identified after a finite number K of iterations, i.e. that v^k lies in T whenever $k \geq L$.

5. Convergence to the equilibrium solution v^* is quadratic, i.e.

$$\limsup_{k\to\infty} \frac{||v^{k+1}-v^*||}{||v^k-v^*||^2} < \infty.$$

Proof. See Dussault and Marcotte [6].

(1) It is assumed that the usual redundant constraints (one for each commodity) of flow conservation equations are deleted.

4. A restriction strategy.

For even moderately sized networks, the linear problem (7) is too large to be solved in a reasonable amount of computer time. Therefore we adopt a restriction (simplicial decomposition) strategy: at each major iteration, indexed by k, algorithm A is used to solve a variational inequality with Φ replaced by the convex hull Φ^k of suitably chosen extreme points $\{y^1,\ldots,y^k\}$ of Φ (see figure 1).

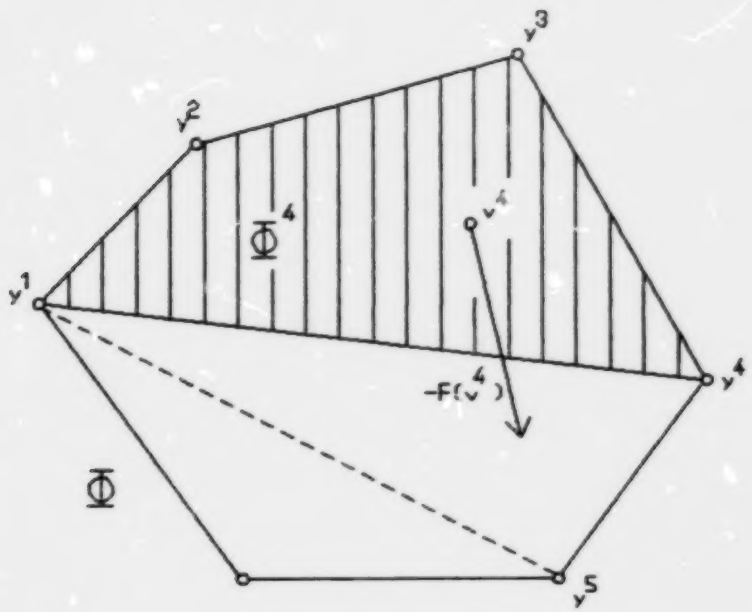

Figure 1. A restriction strategy.

Applied to a minimization problem, the simplicial decomposition strategy generates a sequence of decreasing objective function values. In contrast, in a variational inequality context, no "natural" objective exists;

indeed the restricted gap function is not the same function as the gap function. Thus the gap value can increase from one major iteration to the next. In order to guarantee global convergence of the algorithm, it is therefore necessary to increase the dimension of the master problem's simplex, unless a sufficient decrease in the gap value has occurred. In the following version of the algorithm, all extreme points generated are retained.

Algorithm B.

0. Initialization.

Let y^1 be any extreme point of Φ;

Let $v^1=y^1$ and $\alpha^1 \in \arg\min_{\alpha} b^t\alpha$

$$\text{s.t.} \quad F(v^1) + B^t\alpha \geq 0;$$

$\Phi^1 \leftarrow \{y^1\}$;

$k \leftarrow 1$.

1. Expanding Φ^k.

Let $y^{k+1} \in \arg\min_{v\in\Phi} F^t(v^k)v$;

$\Phi^{k+1} \leftarrow$ convex hull of $\{y^1,\ldots,y^{k+1}\}$.

2. Solving the restricted subproblem.

Let v^{k+1} be solution of the variational inequality:

$$F^t(v^{k+1})(v^{k+1}-v) \leq 0 \quad , \quad \text{for all } v \text{ in } \Phi^{k+1} \tag{B}$$

obtained using algorithm A.

3. Dual variables update.

Let $\alpha^{k+1} \in \arg\min_{\alpha} b^t\alpha$

s.t. $F(v^{k+1}) + B^t\alpha \geq 0$;

$k \leftarrow k+1$ and return to 1.

Any point v in Φ^k can be written as:

$$v = \sum_{i=1}^{k} \lambda_i y^i = Y\lambda$$

where $\lambda=(\lambda_1,\ldots,\lambda_k)^t$ is a positive vector whose components sum to one and $Y=(y^1,\ldots,y^k)$. At step two, the λ_i's can be considered to be the decision variables, and variational problem (8) becomes:

$$G^t(\lambda^{k+1})(\lambda^{k+1}-\lambda) \leq 0 \quad , \quad \text{for all } \lambda \text{ in } S^{k+1} \tag{9}$$

where S^{k+1} denotes the unit simplex in R^{k+1} and $G(\lambda) = Y^tF(Y\lambda)$. The v-solution is then easily reconstructed from $v^{k+1} = Y\lambda^{k+1}$. The gap function associated with variational problem (9) (the restricted gap) is defined, in terms of the vector λ, as:

$$g^{k+1}(\lambda) = \max_{\tau \in S^{k+1}} G^t(\lambda)(\lambda-\tau)$$

The restricted gap is driven to zero by algorithm A by the global minimization of the nonconvex function:

$$\begin{aligned} &\underset{\alpha,\beta}{\text{Min}} \quad G^t(\lambda)\lambda + \beta \\ &\text{s.t.} \quad G(\lambda) + \beta E \geq 0 \\ &\qquad\qquad \lambda \in S^{k+1} \end{aligned} \tag{10}$$

where E denotes the vector $(1,\ldots,1)^t$.

If the vectors $y^1,\ldots,y^{k+1}$ are linearly dependent, then the matrix Y is not of full row rank, and the mapping G may not be strongly monotone. However, it is proved in appendix A that the quadratic convergence rate is not affected by this linear transformation.

At iteration k, a descent direction for (10) is found by solving the linearized problem:

$$\begin{aligned} &\underset{\alpha,\beta}{\text{Min}} \quad [G(\lambda^{k+1}) + G'^t(\lambda^{k+1})\lambda^{k+1}]^t\lambda + \beta \\ &\text{s.t.} \quad G(\lambda^{k+1}) + G'(\lambda^{k+1})(\lambda-\lambda^{k+1}) + \beta E \geq 0 \\ &\qquad\qquad \lambda \in S^{k+1} \end{aligned} \tag{11}$$

5. Solution strategies.

In their implementation of the restriction algorithm, Lawphongpanich and Hearn [12] strived to keep to a minimum the number of extreme points of Φ defining the restricted polyhedron Φ^k. They achieve this goal by deleting the extreme points whose weights in the current solution are small, whenever a sufficient gap decrease has occurred. A similar strategy is adopted here.

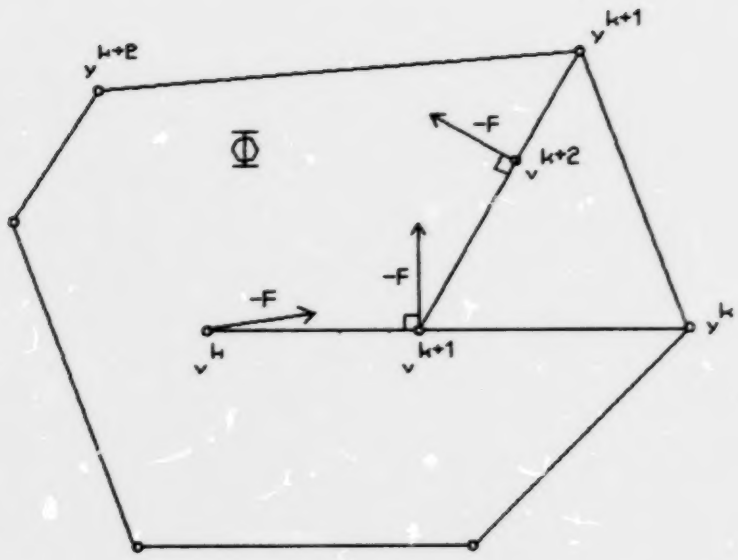

Figure 2. Pseudo Frank-Wolfe algorithm (warm-up iterations).

Also, a very simple and efficient heuristic is applied: in the first few iterations, the cost function is treated as a gradient, and the Frank-Wolfe direction and linesearch are utilized. If F has symmetric Jacobian, this amounts to carrying out regular Frank-Wolfe steps. In the general case, this scheme amounts to solving a one-dimensional variational inequality restricted to the line segment $[v^k, y^k]$, where y^k is a solution to the linear problem

$$\min_{y \in \Phi} F^t(v^k) y.$$

Although this procedure was shown to cycle on small contrived examples

(see Magnanti [13]), the simplex $\sum_{k=1}^{N2} \lambda_k y^k$ obtained after a small number N2 of iterations yields a good starting simplex from which to start the regular algorithm.

Further implementation details are provided in appendix B. Our general algorithm then becomes:

Algorithm C.

0. Initialization.

Let N1 and N2 be predetermined positive integers;

Let ε_1, ε_2 be tolerance factors;

Let y^1 be an extremal point of Φ;

$v^1 \leftarrow y^1$;

$k \leftarrow 1$.

1. Warm-up (memoryless) iterations.

1.1 Solve linearized problem: let $y \in \arg \min_{v \in \Phi} F^t(v^k)v$.

1.2 Solve restricted variational inequality:

$$F^t(v^{k+1})(v^{k+1}-v) \leq 0 \quad , \quad \text{for all } v \in [v^k, y].$$

1.3 $k \leftarrow k+1$;

if $k \leq N1$ goto 1.1, else set k to 1 and goto 2.

2. Getting a good starting restriction.

2.1 Let $y^k \in \arg \min_{v \in \Phi} F^t(v^k)v$.

2.2 Solve restricted variational inequality:

$$F^t(v^{k+1})(v^{k+1}-v) \leq 0 \quad , \quad \text{for all } v \in [v^k, y^k].$$

2.3 k <-- k+1;

if $k \leq N2$ goto 2.1, else goto 3.

3. Regular iterations.

Starting with $\Phi^k = \text{Conv}\{y^1,\ldots,y^{N2}\}$, apply algorithm B until g gets smaller than ε_1. Each major iteration is stopped as soon as the restricted gap becomes less than ε_2. The extreme points with nul weights are dropped out.

6. Numerical results.

The algorithm, in its C version, has been tested on one small-scale and one medium-scale problem.

The former, taken from Nguyen and Dupuis [15], is reproduced in figure 3. The latter corresponds to an assignment for the city of Hull, Canada (501 nodes, 798 links), for which numerical results have also been obtained by Nguyen and Dupuis [15] and Lawphongpanich and Hearn [12].

Our algorithm has been compared, under similar conditions, to the

cutting-plane method of Nguyen and Dupuis and to a diagonalization (Jacobi) relaxation where, at each iteration, the diagonalized problem is solved using PARTAN (see Guélat [8]).

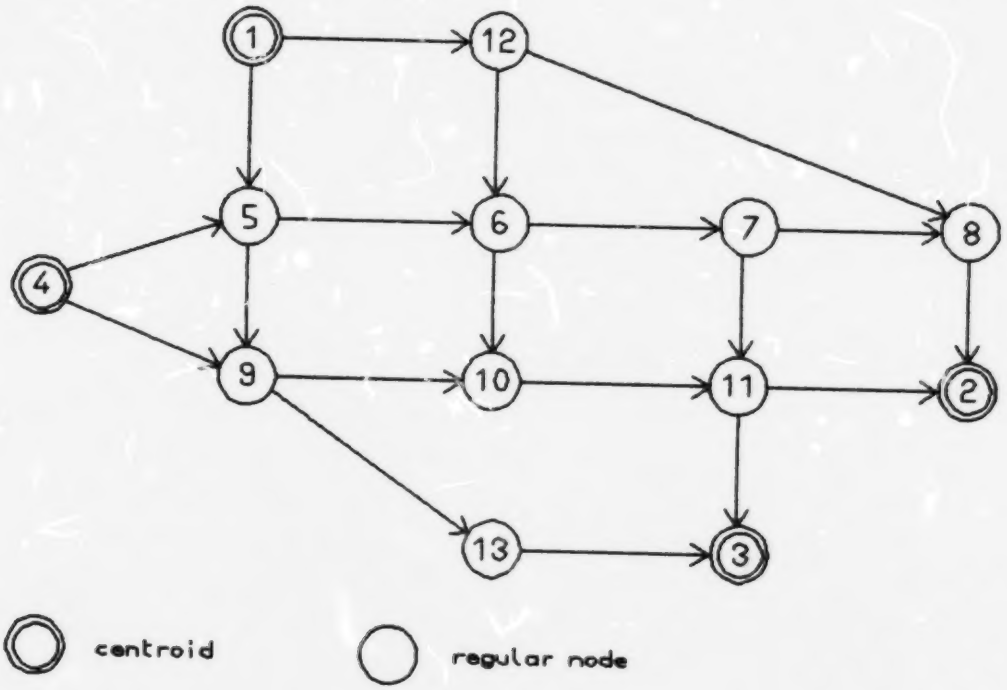

Figure 3. Nguyen and Dupuis network.

5.1 Small-scale problem.

The cost function is affine, $F(v)=Av+b$, with A positive definite. In this case, problem (5) reduces to a convex quadratic program, and the linear approximation method to the standard Frank-Wolfe method.

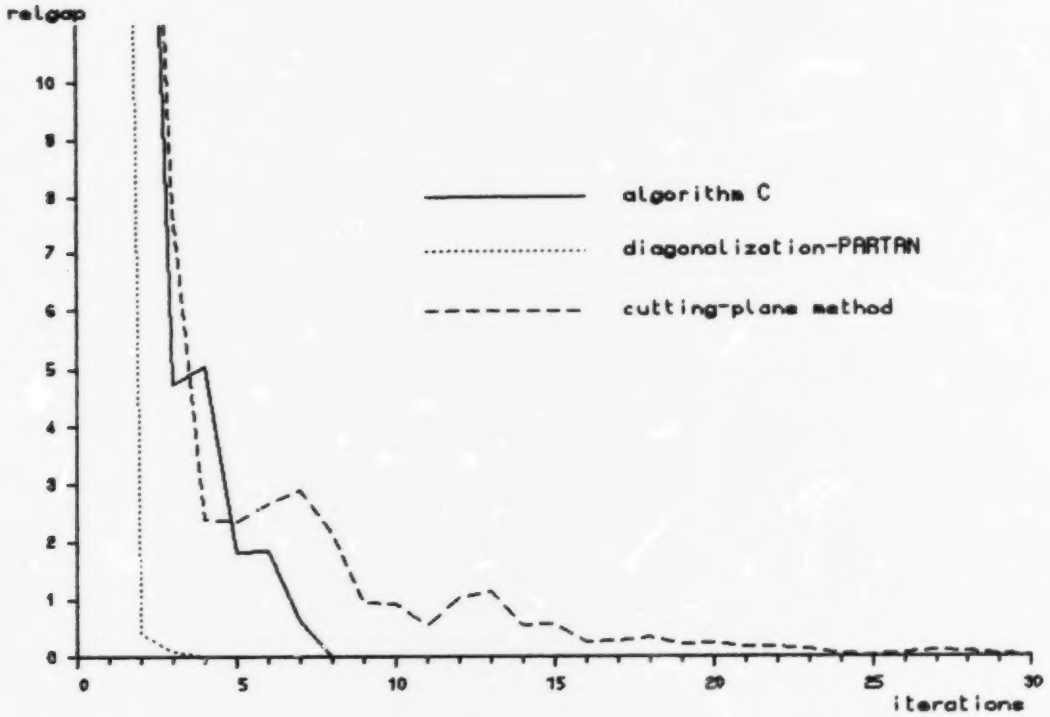

Figure 4. Nguyen and Dupuis network: diagonal cost matrix.

Numerical results for different asymmetry levels are summarized in figures 4,5,6 and 7. The relative gap

$$\text{relgap}(v^k) = \frac{g(v^k)}{F^t(v^k)v^k}$$

is plotted against major iteration index. This measure was considered more adequate than CPU time for comparison purposes, the latter being too small to be meaningful. Each major iteration consists of:

a) algorithm C: solving a restricted variational inequality with tolerance factor $\varepsilon_2 = 10^{-3}$;

b) cutting-plane algorithm: determining a search direction by solving a linear program, and finding a new cut;

c) diagonalization : solving a symmetric problem (with diagonalized costs); the resolution of this problem is stopped if a "symmetric" gap less than 10^{-4} is reached or a maximum of 9 PARTAN iterations are performed [2].

Since the size of the network is actually small, the warm-up iterations of algorithm C are not necessary here, so N1 and N2 are set to zero, and algorithm C reduces to algorithm B.

(2) Theoretically, the diagonalization algorithm should converge in a single major iteration when the matrix A is diagonal. In practice however it can take several iterations to converge, since the maximum number of PARTAN steps in each iteration is limited to 9.

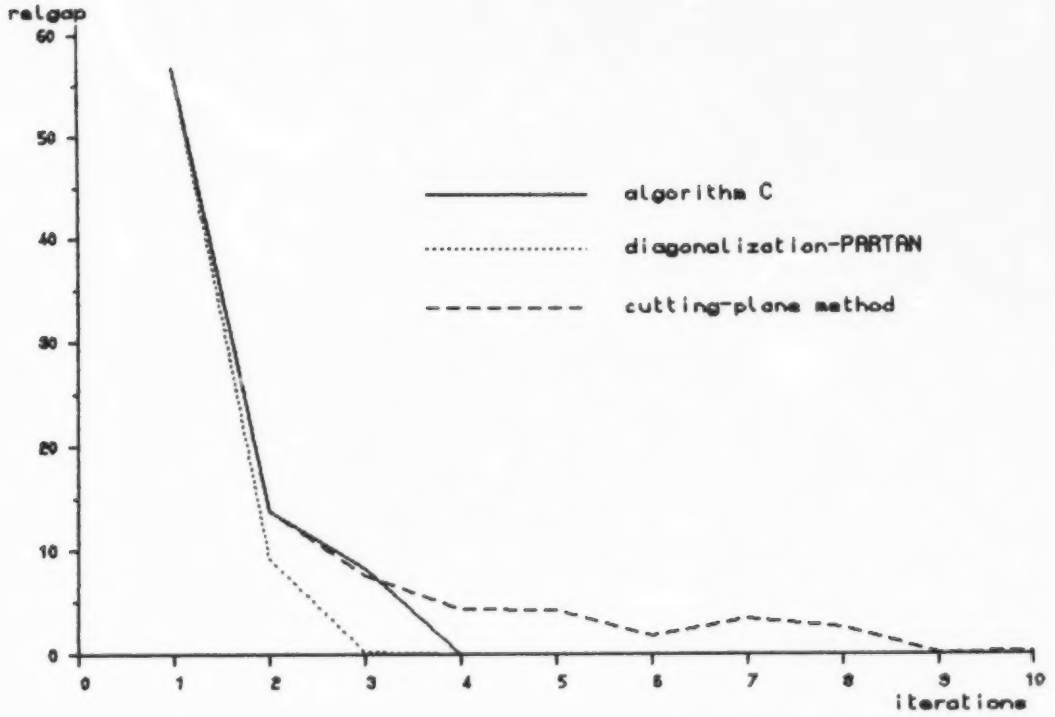

Figure 5. Nguyen and Dupuis network: diagonal dominant cost matrix.

Although not uniformly superior to the other two algorithms, algorithm C's performance is always close to being the best, and is the only one to actually converge in the skew-symmetric case.

On the same problem, Lawphongpanich and Hearn [12] report 8,5 and 6 iterations for the diagonal, diagonal dominant and non diagonal dominant cases, respectively. Their algorithm does not apply to the skew-symmetric case.

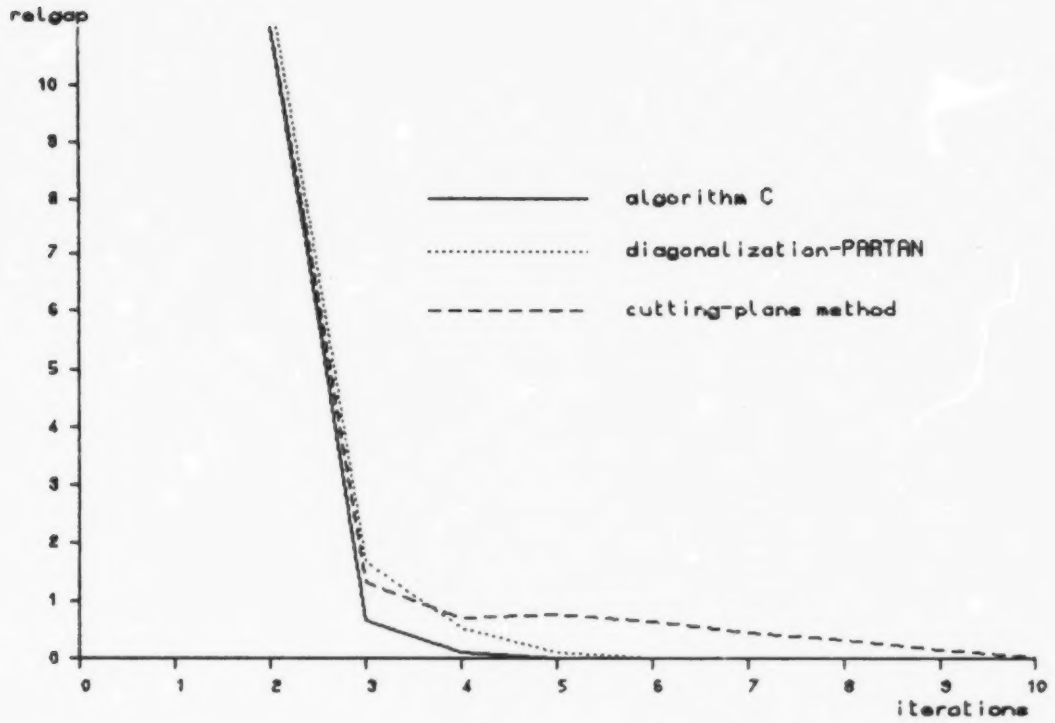

Figure 6. Nguyen and Dupuis network: non diagonal dominant cost matrix.

In the skew-symmetric case (see fig.7), the convergence conditions of the diagonalization and cutting-plane algorithms are no longer satisfied. The sequence produced by the former diverges, whereas the latter converges to a vector which is not an equilibrium flow. On the other hand, skew-symmetry is favorable to algorithm C, making the restricted problem (10) a linear program.

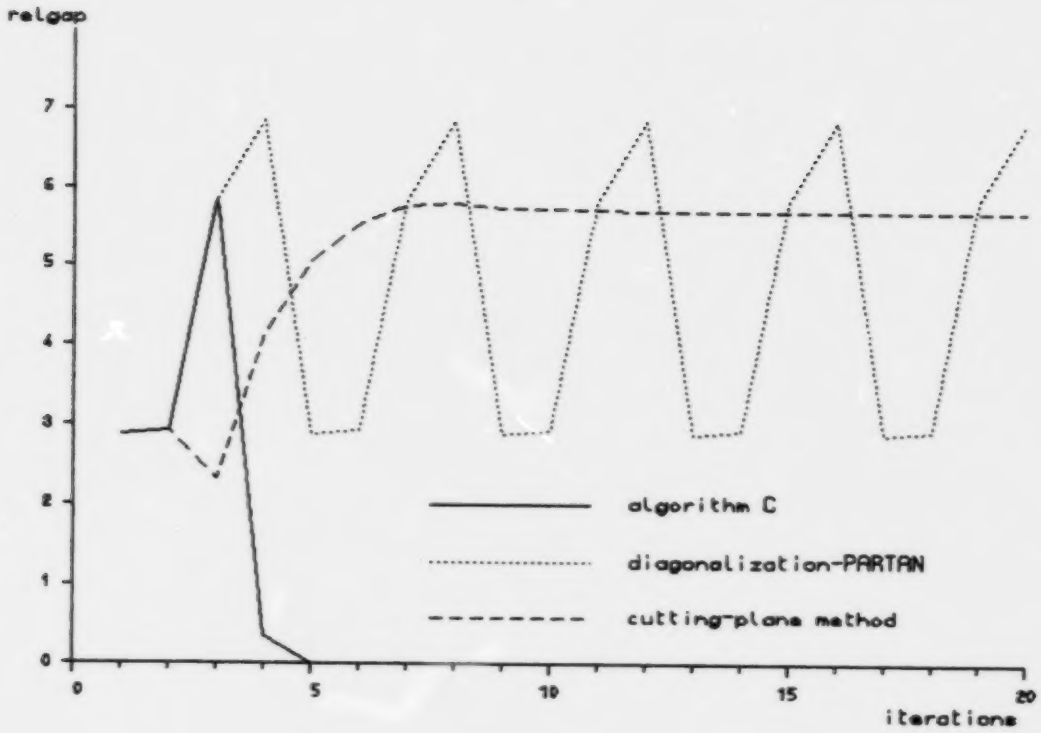

Figure 7. Nguyen and Dupuis network: skew-symmetric cost matrix.

6.2 Medium-scale problem.

The city of Hull problem comprises 501 nodes, 798 links, 23 centroids and 138 origin-destinations pairs. The cost function separates into a non-linear, diagonal part (BPR-type) and a linear, skew-symmetric part, i.e.:

$$H_a(w) = t_0[k_1 + \alpha_a (\frac{v_a}{c_a})^{\beta_a}] + k_2[(A-A^t)w]_a$$

Parameter k_1 insures non-negativity of the cost function. The value zero for parameter k_2 corresponds to the symmetric case, for which

diagonalization-PARTAN reduces to PARTAN and performs very well. However, for even moderate departures from symmetry, the diagonalization algorithm oscillates between extreme points of Φ, and therefore does not converge.

Numerical results are reported in figures 8,9,10 and 11. In this case, CPU-time is more convenient for comparisons than major iteration index, because the work done in each of these iterations may considerably differ from one algorithm to the other.

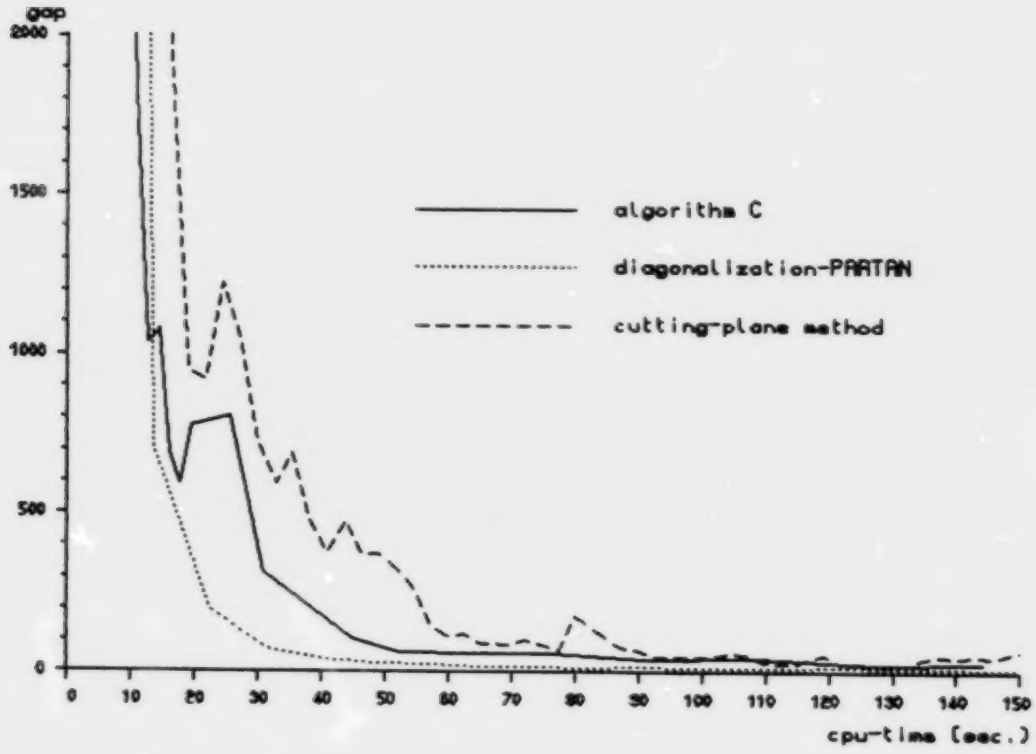

Figure 8. Hull network: $k_1 = 1$, $k_2 = 0$.

Figure 8 shows the results obtained for the symmetric (diagonal) case. Parameters N1 and N2 of algorithm C are both set to 5. In figures 9 and

10, these two parameters are respectively set to 0 and 5. For the last figure, the two values are set to 0.

Although the best results were obtained by setting the parameters to the previous values, setting N1 and N2 to zero yielded good results in all cases.

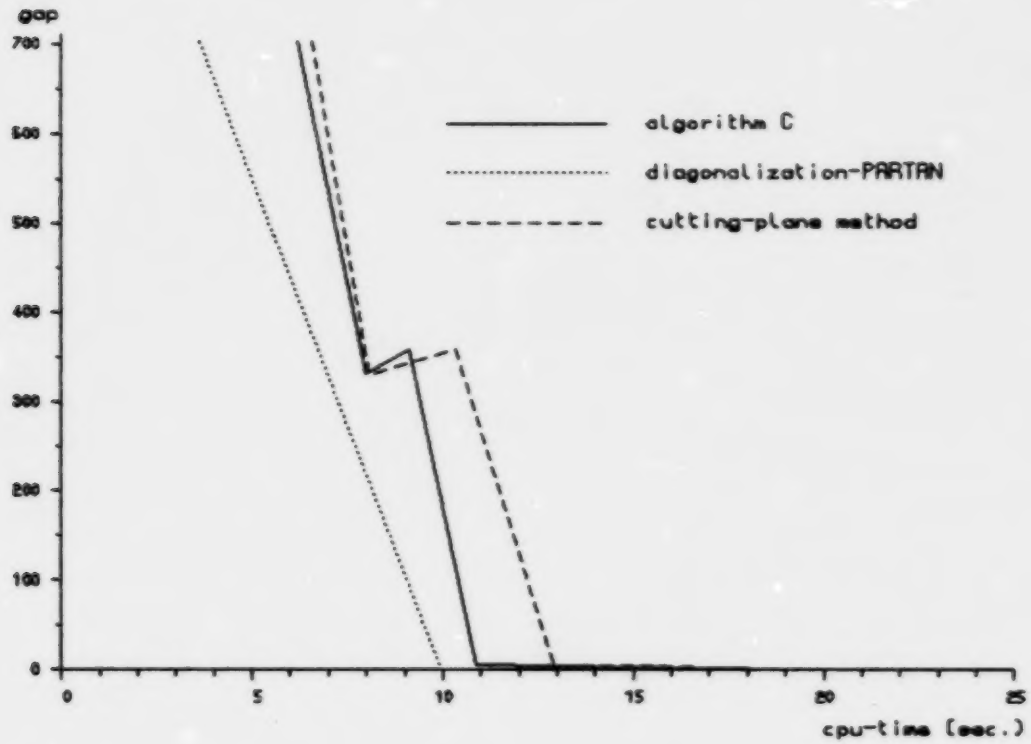

Figure 9. Hull network: $k_1 = 100$, $k_2 = 0.001$.

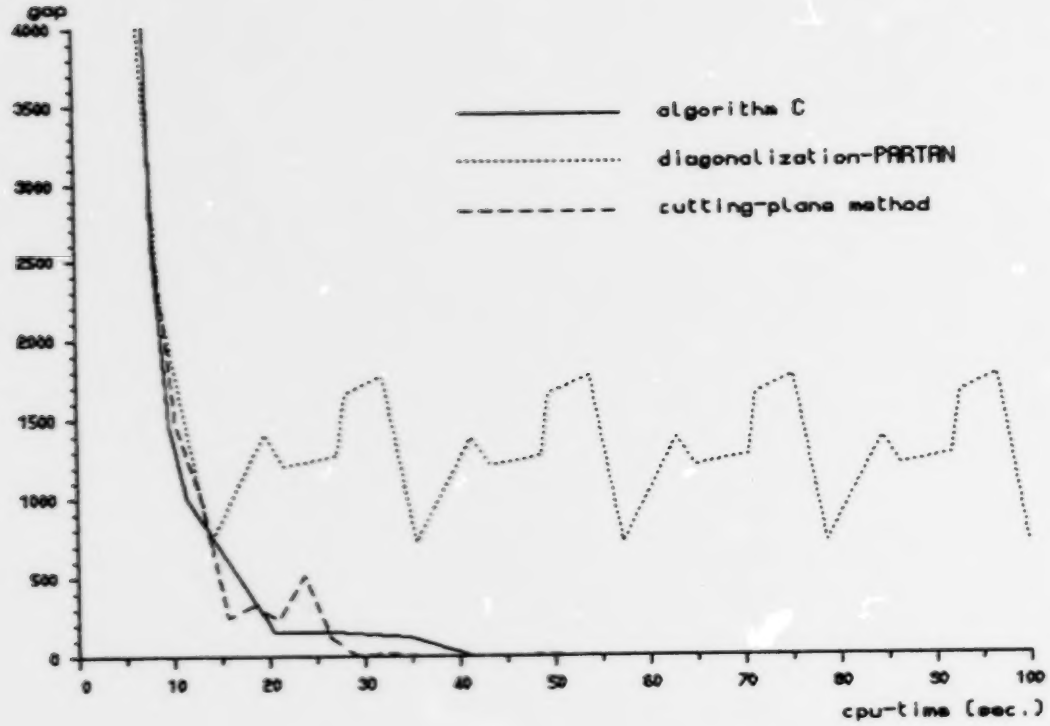

Figure 10. Hull network: $k_1 = 100$, $k_2 = 0.01$.

In all instances, major iterations of algorithm C were stopped whenever the restricted gap became less than $\varepsilon_2 = 10^{-3}$. In the case of the diagonalization algorithm, termination occurred after a maximum of 5 iterations on the symmetrized problem, or when the symmetric gap became less than 10^{-2}.

Again, it can be observed that algorithm C, although not uniformly the best, is always close to being the fastest, the cutting-plane algorithm being a strong contender. The diagonalization scheme, although very efficient on near-symmetric problems, proved incapable of solving strongly

asymmetric test problems.

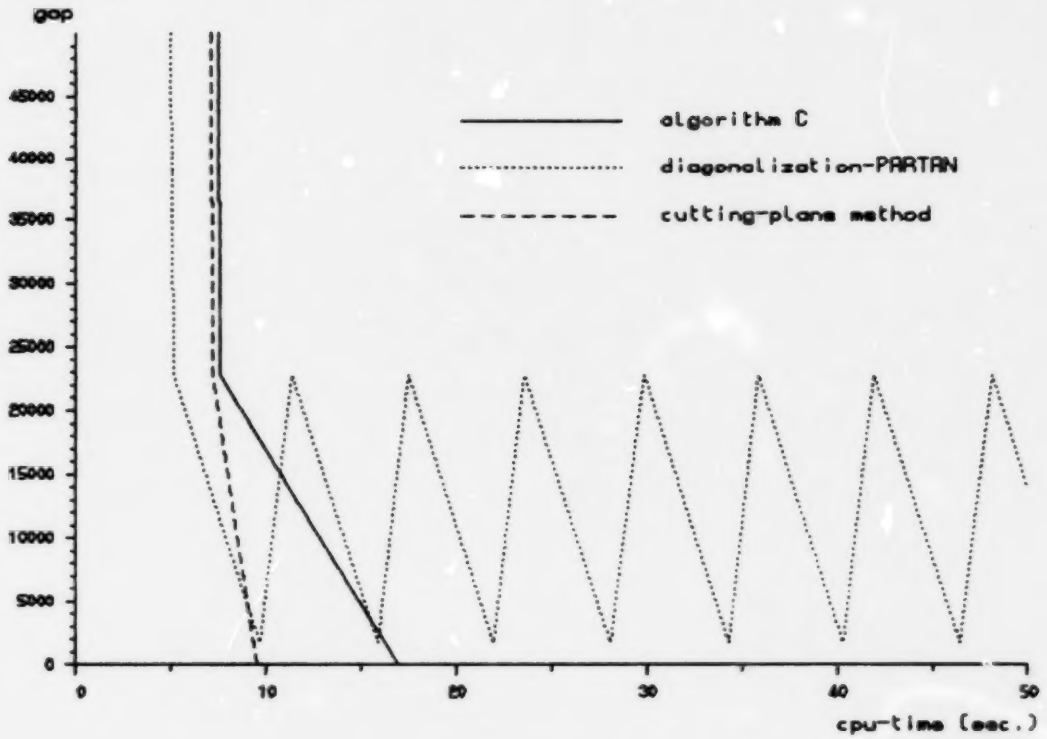

Figure 11. Hull network: $k_1 = 10000$, $k_2 = 1$.

7. Conclusion.

In this paper, we showed how a modified Newton algorithm could effectively solve large-scale network equilibrium problems with asymmetric transportation cost functions. Although slightly more time-consuming than the diagonalization-PARTAN algorithm when applied to symmetric problems, the method proved very robust with respect to asymmetrical perturbations to the cost function.

Appendix A

Quadratic convergence at each major iteration.

In this appendix we show that, if strong monotonicity and strict complementarity in the flow variables hold in the restriction Φ^k, then the quadratic convergence rate of algorithm B is achieved when one utilizes extreme points weights as decision variables.

Proof. Let $y^1,\ldots,y^k$ be the extreme points of Φ^k, $Y=(y^1,\ldots,y^k)$ and $v^k=Y\lambda^k$ the current iterate. Using flow variables, a descent direction for the restricted gap function is extracted from the optimal solution to the linear programming problem:

$$\begin{aligned} &\underset{v,\alpha}{\text{Min}} \quad [F(v^k) + F'^t(v^k)v^k]^t v + b^t\alpha \\ &\text{s.t.} \quad F(v^k) + F'(v^k)(v-v^k) + B_k^t\alpha \geq 0 \qquad (12) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v\in\Phi^k \end{aligned}$$

where B_k is a matrix descriptor of the polyhedron Φ^k, i.e.:

$$\Phi^k = \{v \mid B^k v=b\ ,\ v \geq 0\}$$

In weight-space, the corresponding direction-finding linear program is:

$$\begin{aligned} &\underset{\lambda,\beta}{\text{Min}} \quad [Y^t F(Y\lambda^k) + Y^t F'^t(Y\lambda^k) Y\lambda^k]^t \lambda + \beta \\ &\text{s.t.} \quad Y^t F(Y\lambda) + Y^t F'^t(Y\lambda^k) Y(\lambda-\lambda^k) + \beta E \geq 0 \qquad (13) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda \in S^k \end{aligned}$$

We will show that problems (12) and (13) are equivalent.

Let (v^*,α^*) and (λ^*,β^*) be optimal solutions to (12) and (13) respectively. It is clear that α^* and β^* must be optimal for the following two linear programs:

$$\begin{aligned} &\underset{\alpha}{\text{Min}} \quad b^t\alpha \qquad (14) \\ &\text{s.t.} \quad B_k^t\alpha \geq -[F(v^k) + F'(v^k)(v^*-v^k)] \end{aligned}$$

and:

$$\begin{aligned} &\underset{\beta}{\text{Min}} \quad \beta \qquad (15) \\ &\text{s.t.} \quad \beta E \geq -[Y^t F(Y\lambda^k) + Y^t F'(Y\lambda^k) Y(\lambda^*-\lambda^k)] \end{aligned}$$

The respective dual programs are:

$$\underset{z\in\Phi^k}{\text{Min}} \quad [F(v^k) + F'(v^k)(v^*-v^k)]^t z \qquad (16)$$

and:

$$\underset{\mu\in S^k}{\text{Min}} \quad [Y^t F(Y\lambda^*) + Y^t F'(Y\lambda^k) Y(\lambda^*-\lambda^k)]^t \mu \qquad (17)$$

Clearly, (16) and (17) are equivalent, and the respective optimal values of (14),(15),(16) and (17) are equal.

Let (v^*,α^*) optimal for (12); let $\lambda\in S^k$ such that $v^*=Y\lambda$. Then λ is optimal for (17). If β is set to $b^t\alpha^*$, then β is an optimal dual solution

and (λ,β) is optimal for (13).

Conversely, let (λ^*,β^*) be an optimal solution for (13). Then $v=Y\lambda^*$ is optimal for (16) and, if α is the associated optimal dual solution, then (v,α) is optimal for (12) implying that $b^t\alpha=\beta^*$.

Consequently, there exists a correspondance between optimal solution to problems (12) and (13). Once λ^* is found, algorithm C uses $v^*-v^k = Y(\lambda^*-\lambda^k)$ as search direction. At this point, it is irrelevant whether the algorithm operates in weight-space or in flow-space.

Q.E.D.

The preceding discussion implies that the convergence rate is not affected by the space in which the algorithm operates, although strict complementarity and strong monotonicity <u>cannot</u> be expected to hold in S^k.

Finally, refer to Bertsekas and Gafni [4] for a related convergence rate result when applying a projection algorithm in path-flow space.

Appendix B

Implementation details.

1. Setting-up the direction finding linear problem.

Let, at iteration k, L be the number of extreme points retained in the restriction. A descent direction for the restricted gap function is found by solving a linear programming problem of the form (11), where G and G' are computed using the total arc-flow function H:

$$G(\lambda) = Y^t F(v) = Z^t H(w) = Z^t H(Z\lambda)$$

and $Z^t=(z^1,\ldots,z^L)$ is the matrix whose columns are the total flow vectors corresponding to extremal multicommodity vectors $y^1,\ldots,y^L$ respectively.

Similarly we have:

$$G'(\lambda) = Z^t H'(w) Z$$

The standard simplex tableau corresponding to (11) is given in table 1 below.

$\lambda_1 \ldots \lambda_L \beta_1 \beta_2$	ξ	$e_1 \ldots e_q$	$e_{q+1} \ldots e_L$	η	$t_1 \ldots t_{L-q}$	t	RHS
0	0	0	0	1	1 ... 1	1	0
d^t	1	0	0	0	0	0	0
A_1	0	I_1	0	0	0	0	b_1
$-A_2$	0	0	$-I_2$	0	I_2	0	$-b_2$
E	0	0	0	0	0	1	1

Table 1. Simplex tableau.

$\lambda_1, \ldots, \lambda_L$: primal variables (weights)

$\beta_1 - \beta_2 = \beta$, with $\beta_1, \beta_2 \geq 0$

ξ : objective function value

$e_1, \ldots, e_L$: slack variables

η : phase I objective function value

$t_1, \ldots, t_{L-q}$, t : artificial variables

d^t : cost vector $= (G^t(\lambda^k) + (\lambda^k)^t G'(\lambda^k), 1, -1)$

$$A = \begin{vmatrix} & & & -1 & +1 \\ A_1 & & & \cdot & \cdot \\ & = & -G'(\lambda^k) & \cdot & \cdot \\ A_2 & & & \cdot & \cdot \\ & & & -1 & +1 \end{vmatrix}$$

$E = (1, 1, \ldots, 1, 0, 0)$

I_1, I_2 : unit matrices

$$b = \begin{vmatrix} b_1 \\ \\ b_2 \end{vmatrix} = G(\lambda^k) - G'(\lambda^k)\lambda^k, \quad \text{with } b_1 \geq 0 \ , \ b_2 \leq 0$$

q : dimension of vector b_1.

2. Arc search.

Once a descent direction is found, the gap function is minimized according to Armijo-Goldstein's [2] scheme, starting with a unit step

length. Consequently, when one gets close to the solution, θ can always be set to one, and the method reduces to Newton's method.

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